PRINCIPAL REALIZATION OF THE YANGIAN $Y(\mathfrak{gl}(n))$

CHENG-MING BAI, MO-LIN GE, AND NAIHUAN JING*

Abstract. Motivated to simplify the structure of tensor representations we give a new set of generators for the Yangian $Y(sl(n))$ using the principal realization in simple Lie algebras. The isomorphism between our new basis and the standard Cartan-Weyl basis is also given. We show by example that the principal basis simplifies the Yangian action significantly in the tensor product of the fundamental representation and its dual.

1. Introduction

Loop algebras $L(\mathfrak{g})$ and affine Lie algebras $\hat{\mathfrak{g}}$ are important generalizations of the finite dimensional simple Lie algebra $\mathfrak{g}$. Their structures and representations are usually studied with the help of particular presentations of the Lie algebra. There are two important realizations for the Lie algebra $\mathfrak{g}$ as well as $L(\mathfrak{g})$ and $\hat{\mathfrak{g}}$: the Cartan-Weyl presentation and the principal realization [10]. For example, in the case of $\mathfrak{g} = \mathfrak{gl}(n)$ the Cartan-Weyl basis consists of the usual unit matrices $\{E_{ij}\}_{1\leq i,j\leq n}$ while the principal basis elements $\{A_{ij}\}_{i,j\in\mathbb{Z}/n\mathbb{Z}}$ are Toeplitz matrices, which are diagonal-constant or those in which each descending diagonal from left to right is constant. The principal realization played a pivotal role in constructing the first example of the vertex representation of the affine Lie algebra and opened a new chapter for the conformal field theory.

Yangians are certain quantum deformations of the loop algebras associated to the simple Lie algebras. In the study of the rational solution to the Yang-Baxter equation Drinfeld defined these Hopf algebras as special symmetry algebras [6, 5], and they have been actively studied from various contexts (see the recent comprehensive monograph [13] and also [1, 3, 4, 9]). In [7] V.G. Drinfeld introduced the Yangian for an arbitrary simple Lie algebra as an associative algebra generated by $\{I_\alpha, J_\alpha\}$ which are analog of the loop algebra and then found new realization for the Yangian in [8]. The realization of $Y(\mathfrak{gl}(n))$ via the RTT-equation goes back to Faddeev's school [12]. The generators $\{T_{ij}^{(n)}\}$ in the FRT formulation is usually better for studying properties resembling the matrix operations such

2000 *Mathematics Subject Classification.* 17B37, 17B65, 81R50, 17B67.
Key words and phrases. Yangian, principal realization.
* Corresponding author.

as the quantum determinant [13], while Drinfeld's new basis plays a crucial role in studying finite dimensional representations (cf. [5] and [13]). However the second part of Drinfeld generators called J_α operators are often difficult to compute in various representations due to a symmetric sum in the construction. In [2] an example is found that in certain change of basis the action of the J_α operators seem to be simplified. It is interesting that this example was found in the application of Yang-Baxter R-matrix to maximally entangled states in the context of the newly active field of quantum computation. This raises the question whether there exists another basis fitted for this purpose.

In this paper we shall introduce another basis or presentation for the Yangian $Y(\mathfrak{gl}(n))$ and $Y(\mathfrak{sl}(n))$ using the principal gradation. Our construction is motivated by two reasons. The first is to establish a mathematical model to answer our question in the last paragraph and seek a better way to study representations of the Yangian. There seems to be some interesting properties shared by our new generators and in particular we exploit the additive structure in the index set $I = \{0, 1, \cdots, n-1\}$ by using the discrete Fourier transform on the finite abelian group $\mathbb{Z}_n$. It turns out that in the principal basis some of the actions of the J operators are dramatically simplified, and we offer one example in the last section to support this claim. The second motivation is the connection with quantum computation. We notice that similar construction of the principal basis has been used in several fast algorithms in quantum computation through the discrete Fourier transform. In this aspect we will show by an example that our principal realization indeed has a close relationship with quantum entanglement and it simplifies some of complicated computations experienced in the standard Cartan-Weyl basis for the Yangian.

The paper is organized as follows. Section two reviews some of the basic materials about principal basis for the general linear Lie algebra. Section three discusses how to change the Cartan-Weyl basis to another basis and introduces the principal realization for the Yangian $Y(\mathfrak{gl}(n))$. We also study basic properties for the principal basis and give the isomorphism between the Cartan-Weyl basis and our principal basis. Section four studies the representation of the Yangian in terms of the principal basis. Finally in section five we apply our basis to the tensor product of the fundamental representations of the Yangian $Y(sl(3))$ and show that the principal basis can simplify the action of the Yangian in this case.

2. The general linear Lie algebra $\mathfrak{gl}(n)$

Let $\mathfrak{g} = \mathfrak{gl}(n)$ be the Lie algebra of $n \times n$-complex matrices. The standard unit matrices $\{E_{ij}\}$ form the so-called Cartan-Weyl basis for $\mathfrak{g}$ and

$$[E_{ik}, E_{lj}] = \delta_{kl}E_{ij} - \delta_{ij}E_{lk}, \quad i, j, k, l \in I. \tag{2.1}$$

The enveloping algebra is given by the relation: $E_{ik}E_{lj} = \delta_{kl}E_{ij}$. If we define $\deg(E_{ij}) = j - i$, then $\mathfrak{g}$ becomes a graded Lie algebra

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{g}_i, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \tag{2.2}$$

where $\mathfrak{g}_i = \{x \in \mathfrak{g} | \deg(x) = i\}$. For a matrix $x = (x_{ij})$ we define the principal decomposition:

$$x = x_0 + x_1 + x_2 + \cdots + x_{n-1}, \quad \deg(x_i) = i, \tag{2.3}$$

where $x_i = (x_{k.k+i})_{k,l}$. For instance

$$x_1 = \begin{pmatrix} 0 & x_{12} & 0 & \cdots & 0 \\ 0 & 0 & x_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1,n} \\ x_{n1} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Consider $\sigma \in \mathrm{Aut}(\mathfrak{g})$ defined by

$$\sigma(E_{j,j+1}) = \omega E_{j,j+1}, \quad \sigma(E_{j+1,j}) = \omega^{-1}E_{j+1,j}, \quad \omega = e^{\frac{2\pi i}{n}}, \tag{2.4}$$

and extended to $\mathfrak{g}$ linearly. Then $\mathfrak{g}_i = \{x \in \mathfrak{g} | \sigma(x) = \omega^i x\}$. Let

$$E = \sum_{i=0}^{n-1} E_{i,i+1}$$

then the centralizer $Z(E) = \bigoplus_{i=0}^{n-1} \mathbb{C}E^i$ is a Cartan subalgebra of $\mathfrak{gl}(n)$ called the principal Cartan subalgebra. With respect to the principal Cartan subalgebra, the (principal) root spaces are defined by $\mathfrak{g}_\beta = \{x \in \mathfrak{g} | [h, x] = \beta(h)x, \quad \forall\ h \in Z(E)\}$. In this case the root vectors are given as follows.

For $i \in \mathbb{Z}/n\mathbb{Z}$ let

$$A_i = (\omega^{ki})_{kl} = \begin{pmatrix} \omega^i & \omega^i & \cdots & \omega^i \\ \omega^{2i} & \omega^{2i} & \cdots & \omega^{2i} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{ni} & \omega^{ni} & \cdots & \omega^{ni} \end{pmatrix}$$

and we denote its principal components by A_{ij}:

$$A_i = A_{i0} + A_{i1} + \cdots + A_{i,n-1}, \tag{2.5}$$

i.e. $A_{ij} = (\omega^{ki}\delta_{l-k,j})_{k,l} = \sum_k \omega^{ik} E_{k,j+k}$ for $j \in \mathbb{Z}/n\mathbb{Z}$ and $i = 1, \cdots, n-1$. For completeness we include the Lie algebra structure under the principal decomposition as follows.

$$[E^k, A_i] = (\omega^{ki} - 1)A_i, \quad [E^k, A_{ij}] = (\omega^{ki} - 1)A_{i,j+k}. \tag{2.6}$$

$$[A_{ij}, A_{i'j'}] = (\omega^{ji'} - \omega^{j'i})A_{i+i',j+j'}, \tag{2.7}$$

which follows from the algebra structure: $A_{ij}A_{i'j'} = \omega^{ji'}A_{i+i',j+j'}$.

Under the standard invariant form $(x|y) = \mathrm{tr}(xy)$, the basis elements are isotropic except $n|2i$ and $n|2j$: $(A_{ij}|A_{i'j'}) = n\omega^{-ij}\delta_{i,-i'}\delta_{j,-j'}$.

3. The Yangian $Y(\mathfrak{gl}(n))$

The Yangian $Y(\mathfrak{gl}(n))$ is defined by Drinfeld [7, 12] as the complex associative unital algebra generated by generators $T_{ij}^{(m)}$, $i, j \in \{1, \cdots, n\}, m = 1, 2, \cdots$ subject to the defining relations (cf. [13]):

$$[T_{ij}^{(l+1)}, T_{i'j'}^{(m)}] - [T_{ij}^{(l)}, T_{i'j'}^{(m+1)}] = T_{i'j}^{(l)}T_{ij'}^{(m)} - T_{i'j}^{(m)}T_{ij'}^{(l)}, \tag{3.1}$$

where $l, m = 0, 1, 2\cdots$ and $T_{ij}^{(0)} = \delta_{ij}\cdot 1$. For this reason the generators $T_{ij}^{(m)}$ are Cartan-Weyl type generators. It is well known that $\{T_{ij}^{(1)}\} \cup \{T_{ij}^{(2)}\}$ are enough to span the whole Yangian $Y(\mathfrak{gl}(n))$. The following result can be easily proved using dual bases.

Lemma 3.1 Let $\{e_i\}$ and $\{e^i\}$ be a pair of dual bases of $\mathfrak{g}$. Then the r-matrix can be expressed as follows.

$$r = \sum e_i \otimes e^i. \tag{3.2}$$

Moreover this expression is independent of the choice of the dual bases.

Using the principal basis (2.7) it is easy to get the following.

Corollary 3.2 The permutation matrix can be written as

$$P = \sum_{ij} E_{ij} \otimes E_{ji} = \sum_{k,l} \frac{\omega^{kl}}{n} A_{kl} \otimes A_{-k,-l}.$$

Let u be a formal variable and we consider the operators or matrices in $\mathrm{End}(V\otimes V)[[u^{-1}]]$ as well as $Y(\mathfrak{gl}(n))[[u^{-1}]]$. Let

$$R(u) = I - \frac{P}{u} \in \mathrm{End}(V \otimes V)[[u^{-1}]].$$

The Yang-Baxter equation is an operator equation on $End(V^{\otimes 3})[[u^{-1}]]$ written as

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u),$$

where the indices specify the action of R on respective components of $V^{\otimes 3}$. The so-called T-matrix $T(u) = (T_{ij}(u))$ of the Yangian $Y(\mathfrak{gl}(n))$ can incorporate the defining relations (3.1) into a mater matrix identity as follows. Let

$$T_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} T_{ij}^{(k)} u^{-k} \in Y(\mathfrak{gl}(n))[[u^{-1}]].$$

Then

$$T(u) = \sum_{i.j} T_{ij}(u) \otimes E_{ij} \in Y(\mathfrak{gl}(n)) \otimes \mathrm{End}(V). \tag{3.3}$$

Proposition 3.3 ([7, 8, 12]) The defining relations of the Yangian can be written as

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \tag{3.4}$$

and the coproduct is given by

$$\Delta(T(u)) = T(u) \otimes T(u).$$

We now introduce a new set of generators, which are generalization of the principal generators in $\mathfrak{gl}(n)$. For $k, l \in \mathbb{Z}/n\mathbb{Z}$ we define elements

$$S_{kl}(u) = \sum_{m=0}^{\infty} S_{kl}^{(m)} u^{-m}, \tag{3.5}$$

and $S_{kl}^{(0)} = \delta_{k0}\delta_{l0} \cdot 1$.

We write

$$T(u) = \sum_{k,l \in \mathbb{Z}/n\mathbb{Z}} S_{kl}(u) \otimes A_{kl} \in Y(\mathfrak{gl}(n)) \otimes \mathrm{End}(V).$$

The following result is a standard calculation in FRT formulation.

Theorem 3.4 The principal generators $S_{kl}^{(m)}$ of the Yangian satisfy the following relations:

$$\begin{aligned}[S_{ij}(u), S_{kl}(v)] = \frac{1}{u-v}(&\sum_{a,b} \frac{\omega^{ib-bk-ab}}{n} S_{k+a,l+b}(u)S_{i-a,j-b}(v) \\ &- \sum_{a,b} \frac{\omega^{ja-al+ab}}{n} S_{k+a,l+b}(v)S_{i-a,j-b}(u)).\end{aligned}$$

Proof. From the FRT equation it follows that

$$(I - \frac{P}{u-v})(\sum_{i,j,k,l} S_{ij}(u)S_{kl}(v)A_{ij} \otimes A_{kl}) = (\sum_{i,j,k,l} S_{ij}(v)S_{kl}(u)A_{ij} \otimes A_{kl})(I - \frac{P}{u-v}).$$

Using the principal decomposition of the operator P we have

$$\begin{aligned}
[S_{ij}(u), S_{kl}(v)] &= \frac{1}{u-v}(P\sum_{i,j,k,l}(A_{ij}\otimes A_{kl})S_{ij}(u)S_{kl}(v) - \sum_{i,j,k,l}(A_{ij}\otimes A_{kl})S_{kl}(v)S_{ij}(u)\cdot P) \\
&= \frac{1}{u-v}(\sum_{i,j,k,l,a,b}\frac{\omega^{ab+bi-bk}}{n}(A_{i+a,j+b}\otimes A_{k-a,l-b})S_{ij}(u)S_{kl}(v) \\
&\quad -\sum_{i,j,k,l,a,b}\frac{\omega^{ab+ja-la}}{n}(A_{i+a,j+b}\otimes A_{k-a,l-b})S_{kl}(v)S_{ij}(v)) \\
&= \frac{1}{u-v}\sum_{i,j,k,l,a,b}\frac{\omega^{-ab+bi-bk}}{n}(A_{ij}\otimes A_{kl})S_{k+a,l+b}(u)S_{i-a,j-b}(v) \\
&\quad -\frac{1}{u-v}\sum_{i,j,k,l,a,b}\frac{\omega^{ab+ja-la}}{n}(A_{ij}\otimes A_{kl})S_{k+a,l+b}(v)S_{i-a,j-b}(u).
\end{aligned}$$

□

The defining relations of our principal realization can be written componentwise as follows.

$$\begin{aligned}
[S_{ij}^{(l+1)}, S_{i'j'}^{(m)}] - [S_{ij}^{(l)}, S_{i'j'}^{(m+1)}] &= \sum_{a,b\in\mathbb{Z}/n\mathbb{Z}}\frac{\omega^{(i-i')b-ab}}{n}S_{i'+a,j'+b}^{(l)}S_{i-a,j-b}^{(m)} \\
&\quad -\sum_{a,b\in\mathbb{Z}/n\mathbb{Z}}\frac{\omega^{(j-j')b+ab}}{n}S_{i'+a,j'+b}^{(m)}S_{i-a,j-b}^{(l)}
\end{aligned}$$

where all lower indices are inside $\mathbb{Z}/n\mathbb{Z}$, and $l, m \in \mathbb{N} = \{0, 1, \cdots\}$.

Theorem 3.5. The mapping $T_{ij}(u) \mapsto \sum_k S_{k,j-i}(u)\omega^{ik}$ defines an isomorphism of two presentations of the Yangian $Y(\mathfrak{gl}(n))$. The inverse mapping is given by

$$S_{kl}(u) \mapsto \sum_i \frac{\omega^{-ki}}{n}T_{i+l,i}(u). \tag{3.6}$$

Proof. This can be simply shown by the master equations (3.3-3.4). We prove the inverse isomorphism. Note that $\{A_{kl}\}$ and $\{\frac{\omega^{kl}}{n}A_{-k,-l}\}$ are dual bases.

$$\begin{aligned}
S_{kl}(u) &= (T(u)|A_{-k,-l})\frac{\omega^{kl}}{n} = \sum_{ij}T_{ij}(u)(E_{ij}|A_{-k,-l})\frac{\omega^{kl}}{n} \\
&= \sum_{ij}T_{ij}(u)\omega^{-ki}\delta_{j,-l+i}\frac{\omega^{kl}}{n} = \sum_i\frac{\omega^{k(l-i)}}{n}T_{i,-l+i}(u).
\end{aligned}$$

4. Standard principal representation

Let V be the fundamental representation of $\mathfrak{gl}(n)$ with the standard basis $\{v_i\}_{i=1}^n$. We consider the endomorphism ring $\mathrm{End}(V)$ and $E_{ij} \in \mathrm{End}(V)$:

$$E_{ij}v_k = \delta_{jk}v_i.$$

Using the additive group structure of the index set $\mathbb{Z}/n\mathbb{Z}$, we apply the Fourier transform of the basis element v_i:

$$\varphi_i = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \omega^{ik} v_k. \tag{4.1}$$

Under the natural inner product $(v_i|v_j) = \delta_{ij}$ the vectors φ_i satisfy

$$(\varphi_i|\varphi_j) = \delta_{i+j,0}. \tag{4.2}$$

The transition matrix from the basis $\{\varphi_i\}$ to the basis $\{v_i\}$ is given by the inverse Fourier transform:

$$v_i = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \omega^{-ik} \varphi_k. \tag{4.3}$$

The action of the principal basis is simply

$$A_{kl}\varphi_i = \omega^{il}\varphi_{i+k}.$$

As in the finite dimensional case, the principal realization of the general linear Lie algebra also provides a representation for the Yangian. The following result is a reformulation of the evaluation homomorphism from the Yangian $Y(\mathfrak{gl}(n))$ to the universal enveloping algebra $U(\mathfrak{gl}(n))$.

Proposition 4.1. *The map $\phi(S_{ij}^{(1)}) = A_{ij}$ and $\phi(S_{ij}^{(n)}) = 0$ for $n \geq 2$ gives rise to a representation for the Yangian.*

It is clear that any finite dimensional representation of $U(\mathfrak{gl}(n))$ can also be viewed as a representation of the Yangian as stated in the above proposition.

5. The representations of the Yangian $Y(sl(3))$ and entangled states

In this section, we will show that the principal basis given in the previous sections plays an essential role in the representation theory of the Yangian $Y(sl(3))$ which has a close relation with the study of entangled states [11] in quantum information.

Let λ_1 and λ_2 be the fundamental weights of the simple Lie algebra $sl(3)$. The irreducible representation $V(\lambda_2)$ can be viewed as the dual of the irreducible representation

$V(\lambda_1)$. Suppose $|i\rangle_1 = u_0, u_1$ and u_2 is the standard basis in $V(\lambda_1)$ (quark states), and set $|j\rangle_2 = u_0^*, u_1^*$ and u_2^* are dual base (antiquark states). Note that $V(\lambda_1)$ is isomorphic to the 3-dimensional vector representation. Let $|i,j\rangle = |i\rangle_1 \otimes |j\rangle_2$ $(i,j = 0,1,2)$ be the orthonormal basis in the tensor representation $V(\lambda_1)\otimes V(\lambda_2)$. The $sl(3)$ entangled states with the maximal degree of entanglement were given in [11] as follows.

$$\psi_j^{(i)} = \frac{1}{\sqrt{3}}(|0, i-1\rangle + \omega^{i-1}|1, i\rangle + \omega^{2(i-1)}|2, i+2\rangle), \qquad i, j = 1, 2, 3. \tag{5.1}$$

The Cartan-Weyl basis for the algebra $Y(sl(3))$ are generated by two sets of generators I_α and J_α. In general any finite dimensional irreducible representation V of $sl(3)$ can be lifted to a representation of the Yangian $Y(sl(3))$. In the case of fundamental representations $V(\lambda_i)$ it is given by the homomorphism $J_\alpha = aI_\alpha$ for a fixed constant a, and we denote the resulted presentation by $V(\lambda_i, a)$. When we pass the action of the Yangian $Y(sl(3))$ to the tensor product we need to apply its co-products. According to Drinfeld [8] the finite dimensional irreducible representations are classified by their Drinfeld polynomials, which are determined with help of Drinfeld's new basis; see also [5] and [13] for detailed exposition. We would like to use our principal basis for the tensor product. First of all the Cartan generators are

$$H_1 = E_{11} - E_{22}, H_2 = E_{22} - E_{33}, E_{ij}, i \neq j \tag{5.2}$$

to give the action of the Yangian $Y(sl(3))$ on the tensor product $V(\lambda_1, a) \otimes V(\lambda_2, b)$. We will give the action of the Yangian operators $J(H_1), J(H_2), J(E_{ij})$ on the standard basis of the representations of the Lie algebra $sl(3)$ (given by Clebsch-Gordon coefficients) according to the following decomposition of the representations of $sl(3)$

$$V(\lambda_1) \otimes V(\lambda_2) \cong V(\lambda_1 + \lambda_2) \oplus V(0). \tag{5.3}$$

If we use the standard basis $|i,j\rangle$ the action of the Yangian is very complicated and it is not obvious to see whether there exist certain rules.

We use the entangled states $\psi_j^{(m)}$ given by Eqs. (5.1)-(5.3) as the basis for the representation space, and slightly modify the principal basis A_{ij} $(ij \neq 00)$ as follows.

$$T_i^{(j)} = \omega^{3-i+1}A_{i-1,j-1}, \ \ i, j = 1, 2, 3 \ \text{ and for } \ j = 1, i \neq 1. \tag{5.4}$$

Then comparing with the basis given by Eq. (5.2), we know that

$$T_2^{(1)} = H_1 - \omega^2 H_2, \ \ T_3^{(1)} = H_1 - \omega H_2,$$

$$T_1^{(2)} = E_{12} + E_{23} + E_{31},\ T_2^{(2)} = E_{12} + \omega E_{23} + \omega^2 E_{31},\ T_3^{(2)} = E_{12} + \omega^2 E_{23} + \omega E_{31},$$
$$T_1^{(3)} = E_{13} + E_{21} + E_{32},\ T_2^{(3)} = E_{13} + \omega E_{21} + \omega^2 E_{32},\ T_3^{(3)} = E_{13} + \omega^2 E_{21} + \omega E_{32}. \quad (5.5)$$

In terms of Eq. (5.5) the Yangian operators $J(T_2^{(1)}), J(T_3^{(1)}), J(T_1^{(2)}), \cdots$ can be defined according to the action of Yangian operators expressed in terms of H_1, H_2 and $E_{ij}(i \neq j)$.

A direct computation proves the following result.

Theorem 5.1 The action of the Yangian $Y(sl(3))$ on the tensor product of $V(\lambda_1, a) \otimes V(\lambda_2, b)$ can be expressed in the action of the principal basis $J(T_i^{(j)})$ on the entangled states $\psi_k^{(m)}(k, m = 1, 2, 3)$ with the maximal degree of entanglement in a simple way. The explicit action is given by the following equation:

$$J(T_i^{(j)})\psi_k^{(m)} = [a\omega^{(j-1)(k-1)} - b\omega^{(i-1)(m-1)} + \frac{3}{2}\delta_{i+k-1,1}\delta_{m+j-1,1}\omega^{(j-1)(k-1)} - \frac{3}{2}\delta_{k,1}\delta_{m,1}]\psi_{i+k-1}^{(m+j-1)}.$$

We remark that the vectors $\psi_k^{(m)}$ provide a nice basis which simplifies the action of $Y(sl(3))$ dramatically, and the transition operators for the entangled states $\psi_k^{(m)}(k, m = 1, 2, 3)$ are with the maximal degree of entanglement in quantum computation.

From Theorem 5.1 it is easy to get the following conclusion ([5], [13]).

Corollary 5.2 Let $W = V(\lambda_1, a) \otimes V(\lambda_2, b)$. If $|a - b| \neq \frac{3}{2}$, then W is an irreducible representation of the Yangian $Y(sl(3))$. Otherwise, W has a unique proper $Y(sl(3))$-subrepresentation V given as follows.

(1) If $a - b = \frac{3}{2}$, we have $V \cong V(0, 0)$, and $W/V \cong V(\lambda_1 + \lambda_2)$ as vector spaces.

(2) If $a - b = -\frac{3}{2}$, we have $V \cong V(\lambda_1 + \lambda_2)$ as vector spaces.

Acknowledgments

Jing is grateful to the support of NSA grant H98230-06-1-0083 and NSFC's Overseas Distinguished Youth Grant (10728102). This work was supported in part by the National Natural Science Foundation of China (10575053, 10571091, 10621101), NKBRPC (2006CB805905), Program for New Century Excellent Talents in University.

References

[1] Arnaudon, D., Molev, A., Ragoucy, E., "On the R-matrix realization of Yangians and their representations", Annales Henri Poincare, **7** , 1269–1325 (2006).

[2] Bai, C. M., Ge, M. L., Xue, K., "Yangian and Its Applications", Inspired by S.S. Chern: A Memorial Volume in Honor of A Great Mathematician, Edited by P.A. Griffiths, 45-93, World Scientific Pub., Singapore, 2006.

[3] Briot, C., Ragoucy, E., "RTT presentation of finite W-algebras", J. Phys. A **34**, 7287–7310 (2001).

[4] Brundan, J., Kleshchev, A., "Parabolic presentations of the Yangian $Y(gl_n)$", Comm. Math. Phys. **254**, 191–220 (2005).

[5] Chari, V., Pressley, A., "A guide to Quantum Groups", Cambridge Univ. Press, Cambridge, 1994.

[6] Cherednik, I., "Quantum groups as hidden symmetries of classical representation theory", in: Differential geometric methods in theoretical physics (Chester, 1988), pp. 47–54, World Sci. Publishing, 1989.

[7] Drinfeld, V., "Hopf algebras and the quantum Yang-Baxter equation", Soviet Math. Dokl. **32**, 254–258 (1985).

[8] Drinfeld, V., "A new realization of Yangians and quantized affine algebras", Soviet Math. Dokl. **36**, 212–216 (1988).

[9] Iohara, K., "Bosonic representations of Yangian double DY(g) with g = gl_N, sl_N", J. Phys. A. **29**, 4593–4621 (1996).

[10] Kac, V. G., *Infinite dimensional Lie algebras*, 3rd ed., Cambridge Univ. Press, 1990.

[11] Kaszlikowski, D., Oi, D.K.L., Christandl, M., Chang, K., Ekert, A., Kwek, L.C., Oh, C.H., Phys. Rev. B 67 012310 (2003).

[12] Kulish, P. P. , Slyanin, E. K., *Quantum spectral transform method: recent developments*, in Integrable Quantum Field Theories, Lect. Notes in Phys. **151**, Springer, Berlin, 1982, pp. 61-119.

[13] Molev, A. I., *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, **143**. American Mathematical Society, Providence, RI, 2007.

Bai: Chern Institute of Mathematics and LPMC, Nankai University, Tanjin 300071, China

E-mail address: baicm@nankai.edu.cn

Ge: Chern Institute of Mathematics, Nankai University, Tanjin 300071, China

E-mail address: geml@nankai.edu.cn

Jing: Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, U.S.A.

E-mail address: jing@math.ncsu.edu